Early termination criterions for polyhedral enumeration

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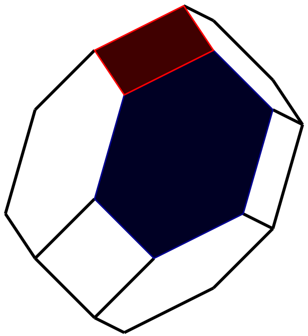
Polytopes, definition

- A polytope $P \subset \mathbb{R}^n$ is defined as:

$$P = \{x \in \mathbb{R}^n \mid f^i(x) \geq b_i \text{ with } f_i \text{ linear}\}$$

  The f_i define facets of the polytope.



- The vertices are points that are not the sum of two points of the polytope.
- The skeleton graph is the graph defined by the adjacency between vertices.

The dual description problem

- Given a polytope P defined by its facets, we want to find its vertices.
- Example. The metric polytope MET_n is defined by the function d_{ij} for $1 \leq i, j \leq n$ satisfying
  - $d_{ij} = d_{ji}$
  - $d_{ij} \leq d_{ik} + d_{kj}$.
  - $d_{ij} + d_{jk} + d_{ki} \leq 2$
- The symmetry group has size $2^{n-1} n!$ and there is a special class of vertices named metric cut.
- The polytope MET_8 has 1550825600 vertices in 533 orbits.
- **Conjecture** (Laurent-Poljak): If $n \geq 6$ if we remove the metric cuts, then the skeleton graph of MET_n remains connected.

Computing dual description of polytopes with symmetry is a quite important problem.

The adjacency decomposition method

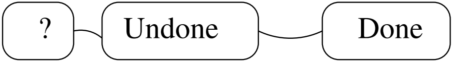
Input: The facet-set of a polytope P and a group G acting on P.
Output: $\mathcal{O}$, the orbits of vertices of P.

- Compute some initial vertex v (by linear programming) and insert the corresponding orbit into $\mathcal{O}$ as undone.
- For every undone orbit O of vertex:
  - Take a representative v of O.
  - Find the direction of edges in the local cone (this is a dual description computation).
  - For every direction, find the corresponding vertex.
  - For every adjacent vertex test if it is equivalent to one already in the list and insert it if needed.
  - Mark the orbit O as done.
- Terminate when all orbits are done.

Reinvented many times (D. Jaquet 1993, T. Christof and G. Reinelt 1996).

How running the algorithm looks like

- We start with some vertex and at the beginning we found many other orbits of vertices.
- Then as the enumeration goes, it levels off and fewer orbits are found.
- Until no new orbits are found, but the enumeration is not finished.
- Most of the runtime is in the last few orbits of high incidence and usually high symmetry.
- What could be done about those pesky undone orbits?



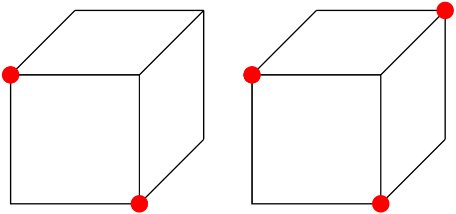
- Note: Sometimes some new orbits are found after the highest incidence one is treated. Example: Perfect domain of E_7 and contact polytope of $\mathbf{O}_{23}$.

Termination criterions

- The problem is that the computation of the adjacent vertex is itself a dual description problem.
- There are ways to deal with that like using the method recursively.
- But we concentrate here on geometrical criterion that allow to terminate early the enumeration.
- In the following we denote $\mathbf{V}_{undone}$ the set of undone vertices.
- We needs to prove is that $\mathbf{V}_{undone}$ is not a cutset in the skeleton graph.

The Balinski theorem and how to use it

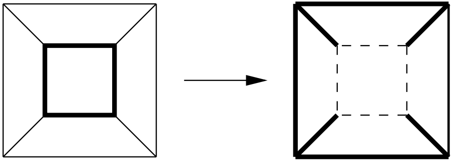
- Theorem: The skeleton graph of a n-dimensional polytope is n connected.
(That is the removal of any $n-1$ vertices leaves it connected)



- So, if $|\mathbf{V}_{undone}| \leq n-1$ then we can stop the enumeration prematurely.
- This is actually quite helpful and allow to prematurely terminate some enumeration procedures. This is because we often work in high dimension.

The linear programming criterion

- Take a facet F of a polytope and remove all edges contained in F. Then the resulting graph is still connected.



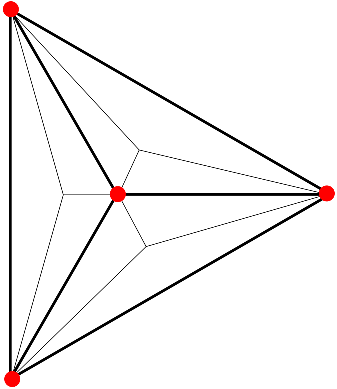
- So, if $\mathbf{V}_{undone}$ is contained in a facet F of the polytope then we can terminate the enumeration prematurely.

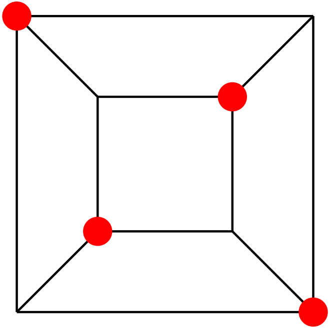
Improved Balinski theorems

- The proof of Balinski theorem uses that among $n-1$ points $\{v_1, \dots, v_{n-1}\}$ in dimension n we have a pencil of hyperplane containing them.
- From the hyperplanes we can optimize by linear programming and prove the connectedness that way.
- Improved Balinski Theorem: If $\{v_1, \dots, v_m\}$ points in dimension n are contained in an affine space of dimension $n-2$ then removing them does not disconnect the graph.
- So, if $\dim Span(\mathbf{V}_{undone}) \leq n-2$ then we can terminate.
- This theorem has actual applicability.
- Another improvement would be:
  If $\{v_1, \dots, v_m\}$ span a space H of dimension $n-1$ and there is another vertex contained in H then removing them does not disconnect the graph.

Dead ends

- We could analyse the structure of the group. In particular getting the representation theory of the group is much simpler than the dual description of the polytope.
  But how could we use that information?
- For the set of undone vertices, we can compute their adjacencies relatively easily.
- But is that information usable? It would seem no.





Recursive approach

- Suppose that for each facet we have proved that for each facet F, $F \cap \mathbf{V}_{undone}$ is not a cutset, then $\mathbf{V}_{undone}$ is not a cutset.
- This idea can be exploited relatively easily.
- Now suppose that for some facet this does not work:
  - Then we can compute the facets of the facets by linear programming.
  - And we can apply the test recursively.
- Also, if we prove the result for all facets except 1 then it is not a cutset.

The problem is that we do not have a single example where this approach works.

Availability

All the relevant software are available at

`https://github.com/MathieuDutSik/polyhedral_common`